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Baxter's quantum number in the XY model

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Abstract. In the particular case of the asymmetric XY model an explicit form for the operator corresponding to Baxter's quantum number n is obtained. It is shown that this operator is closely associated with a transformation of spin operators which converts the asymmetric XY hamiltonian into a symmetric XY hamiltonian with alternating nearest-neighbour interactions.

1. Introduction

In its simplest form the XY model in one dimension consists of a chain of spins one-half interacting by nearest-neighbour forces described by the simple hamiltonian

$$\mathscr{H}_{XY} = -\tfrac{1}{4}\sum_{j=1}^{N}[(1+k)\sigma_j^x\sigma_{j+1}^x+(1-k)\sigma_j^y\sigma_{j+1}^y], \tag{1.1}$$

where the σ_j are Pauli matrices with periodic boundary conditions $\sigma_{j+N} = \sigma_j$. This model has a well known algebraic solution (Lieb *et al* 1961, Katsura 1962) in terms of fermion quasi-particle operators. The model happily arises also as one of the simplest limiting cases (Jones 1973) of Baxter's solution of the general XYZ model (Baxter 1972, 1973a, b, c). The solution due to Baxter is characterized by a new integer-valued quantum number which is different from the total fermion number in the usual solution of this model. In our previous study of the model we achieved an algebraic form of Baxter's solution but were unable to find the operator corresponding to his new quantum number. In the present work we will show that by a slightly different choice of the Baxter basis states we can explicitly write down this operator. Further, we find a spin transformation which, apart from end effects, turns the asymmetric XY model into a symmetric XY model in which the nearest-neighbour interaction strength alternates along the chain of spins.

Thus in § 2 we describe the change in the basis states of our earlier work and construct the Baxter 'spin' operator R_z. In § 3 we define a transformation from σ-spin to ρ-spin and re-express both $\mathscr{H}_{XY}$ and R_z. In § 4 we briefly indicate the connection with the familiar solution in terms of fermion quasi-particles. Throughout we will use notation borrowed either from Baxter's work or from our earlier work on the XY model and the reader is referred there for fuller explanation.

2. The Baxter quantum number

Baxter's diagonalization of the eight-vertex model transfer matrix (Baxter 1973a, b, c) utilizes remarkable families of vectors characterized by a number n of down 'spins' where Baxter's 'spins' are defined as being either 'up' or 'down' with respect to an axis that rotates from site to site along the spin chain. Normalized spinors on each site may be constructed in terms of a function $p(l, s)$ where l is an integer that varies from site to site and s is a free parameter. In our earlier study of the XY model (Jones 1973), we used a form of $p(l, s)$ which had the advantage of giving real basis spinors which were linearly independent in the regime defined by $0 < k \leqslant 1$ but the disadvantage that when $k = 0$ the basis states used ceased to be linearly independent. If we modify the choice of $p(l, s)$ so that the basis spinors are no longer real, we can then obtain an orthonormal basis for the XY model which remains orthonormal at $k = 0$ and there coincides with the states of the XXZ model also studied earlier (Jones 1974).

Thus our new choice for $p(l, s)$ is

$$p(l, s) = \sqrt{k}\,\mathrm{sn}(s+K+2l\eta-\tfrac{1}{2}\mathrm{i}K'), \tag{2.1a}$$

where sn in the Jacobi elliptic function of modulus k, K and K' are the complete elliptic integrals of the first kind of modulus k and complementary modulus $k' = (1-k^2)^{1/2}$ respectively, l is an integer, s is a real parameter, and η is of the form (Baxter 1973a)

$$\eta = \frac{2m_1}{L}K, \tag{2.1b}$$

with m_1, L integers. After expressing sn in terms of jacobian theta functions, it is straightforward to show that

$$\lim_{k\to 0} p(l, s) = \exp[\mathrm{i}(s+2l\eta)], \tag{2.2}$$

where $\exp[\mathrm{i}(s+2l\eta)]$ is the form of $p(l, s)$ appropriate to the XXZ model (Jones 1974). By using the behaviour of sn under translation of its argument by $\mathrm{i}K'$ together with its real analyticity and its addition theorem, one may establish that

$$|p(l, s)| = 1, \tag{2.3a}$$

$$p(l, s) = \mathrm{e}^{\mathrm{i}\delta(l,s)}, \tag{2.3b}$$

where

$$\tan\delta(l, s) = -\frac{\mathrm{cn}(s+K+2l\eta)\,\mathrm{dn}(s+K+2l\eta)}{(1+k)\,\mathrm{sn}(s+K+2l\eta)}. \tag{2.3c}$$

Since $p(l, s)$ is now a complex number of modulus one, we choose the basic 'up' spinor at site j to be

$$\phi_{l_j,l_j+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} p(l_j, s) \\ 1 \end{pmatrix}, \tag{2.4a}$$

and the 'down' spinor to be

$$\phi_{l_j,l_j-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -p^*(l_j, s) \end{pmatrix}. \tag{2.4b}$$

We specialize to the XY model by taking $L = 4, m_1 = 1$ in (2.1b) to give $2\eta = K$. With this value of η one may easily show that

$$\tan\delta(l, s)\tan\delta(l+1, s) = -\left(\frac{1-k}{1+k}\right), \tag{2.5a}$$

and

$$\delta(l+2, s) = \delta(l, s)+\pi. \tag{2.5b}$$

For most values of s, $\delta(l, s)$ depends on the magnitude of the asymmetry parameter k. However, if s is an integer multiple of K, then the phase δ becomes independent of k. Thus later we will set $s = 0$ to simplify our manipulations, for with $s = 0$ we find the simple result

$$\delta(l, 0) = \tfrac{1}{2}l\pi, \tag{2.5c}$$

valid for all values of k. This does not contradict the relation (2.5a) if the left-hand side of that equation is treated carefully in taking the limit $s \to 0$.

If we now form suitable direct products of the spinors (2.4a, b), we obtain the Baxter basis states

$$\psi(l_1, l_2, \ldots, l_N, l_{N+1}) = \phi_{l_1,l_2}\otimes\phi_{l_2,l_3}\otimes \ldots \otimes\phi_{l_N,l_{N+1}} \tag{2.6a}$$

where the integers l_j are constrained by

$$l_{j+1} = l_j \pm 1, \tag{2.6b}$$

$$l_{N+1} \equiv l_1 (\mathrm{mod}\ L). \tag{2.6c}$$

These states may also be written

$$\psi(l_1, l_2, \ldots, l_N, l_{N+1}) = \psi(l; x_1, \ldots, x_n) \tag{2.6d}$$

where $l = l_1$ and $x_1 < x_2 < \ldots < x_n$ label spin sites along the chain at which a 'down' spinor occurs in the product defining $\psi(l; x_1, \ldots, x_n)$. For the XY model ($L = 4$), $\psi(l + 4; x_1, \ldots, x_n) = \psi(l; x_1, \ldots, x_n)$ giving basis states for $l = 1, 2, 3, 4$. The number n of 'down' spinors is constrained by

$$n \equiv \tfrac{1}{2}N (\mathrm{mod}\ \tfrac{1}{2}L), \tag{2.7}$$

where we are now assuming the total number of spin sites N to be even. Thus in the XY model ($L = 4$) n takes either even or odd integer values depending upon whether N is divisible by four or only by two. As shown earlier (Jones 1973) a complete set of states is obtained by taking $\psi(l; x_1, \ldots, x_n)$ and $\psi(l+2; x_1, \ldots, x_n)$ for all allowed values of n and all choices of $x_1 < x_2 < \ldots < x_n$. These states are normalized so that

$$(\psi(l; x_1, \ldots, x_n), \psi(l; y_1, \ldots, y_m)) = \delta_{nm}\delta_{x_1,y_1}\delta_{x_2,y_2} \ldots \delta_{x_n,y_n}, \tag{2.8a}$$

and

$$(\psi(l; x_1, \ldots, x_n), \psi(l+2; y_1, \ldots, y_m)) = 0. \tag{2.8b}$$

In our earlier study using real basis vectors the scalar product (2.8b) did not vanish in all instances. With our new choice of $p(l, s)$ the scalar products (2.8a, b) are independent of k and we thus obtain a complete orthonormal basis for all values of k.

Let us specialize even further by using as a basis the states $\psi(l; x_1, \ldots, x_n)$ in which l is an odd integer ($l = 1, 3$). With this choice and noting the constraints (2.6b, c) we observe that

$$l_j \equiv j (\mathrm{mod}\ 2), \tag{2.9a}$$

and hence

$$p(l_j, s) = \pm p(j, s), \tag{2.9b}$$

where the upper sign holds if $l_j \equiv j(\text{mod } 4)$ and the lower sign if $l_j \not\equiv j(\text{mod } 4)$. Keeping this observation in mind, define an operator on the jth site by

$$u_j(s) = \cos\delta(j,s)\sigma_j^x - \sin\delta(j,s)\sigma_j^y,$$
$$u_j(s) = \begin{pmatrix} 0 & e^{i\delta(j,s)} \\ e^{-i\delta(j,s)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & p(j,s) \\ p^*(j,s) & 0 \end{pmatrix}. \tag{2.10}$$

Using (2.4), (2.9b) and (2.10) one sees that

$$\begin{aligned} u_j(s)\phi_{l_j,l_j+1} &= \pm\phi_{l_j,l_j+1}, \\ u_j(s)\phi_{l_j,l_j-1} &= \mp\phi_{l_j,l_j-1}, \end{aligned} \tag{2.11}$$

where again the upper sign in (2.11) is to be taken if $l_j \equiv j(\text{mod } 4)$ and the lower sign otherwise. Finally, from (2.11) we can compute that

$$u_j(s)u_{j+1}(s)\phi_{l_j,l_{j+1}}\otimes\phi_{l_{j+1},l_{j+2}} = \pm\phi_{l_j,l_{j+1}}\otimes\phi_{l_{j+1},l_{j+2}}, \tag{2.12a}$$

for $j = 1, \ldots, N-1$ and also that

$$u_N(s)u_{N+1}(s)\phi_{l_N,l_{N+1}}\otimes\phi_{l_1,l_2} = \pm\phi_{l_N,l_{N+1}}\otimes\phi_{l_1,l_2}, \tag{2.12b}$$

where we take the $+(-)$ sign if $\phi_{l_{j+1},l_{j+2}}$ or ϕ_{l_1,l_2} is an 'up' ('down') spinor. If we define

$$R_z(s) = \tfrac{1}{2}\sum_{j=1}^{N} u_j(s)u_{j+1}(s), \tag{2.13a}$$

we have that for odd integer values of l,

$$R_z(s)\psi(l; x_1, \ldots, x_n) = (\tfrac{1}{2}N - n)\psi(l; x_1, \ldots, x_n). \tag{2.13b}$$

It is tedious but straightforward to check that $R_z(s)$ commutes with $\mathscr{H}_{XY}$ for any value of the parameter s. The method of proof is to show that there is an operator $O_j(s)$ such that

$$[\mathscr{H}_{XY}, u_j(s)u_{j+1}(s)] = O_{j+1}(s) - O_j(s), \tag{2.14}$$

from which one obtains at once

$$[\mathscr{H}_{XY}, R_z(s)] = \sum_{j=1}^{N} (O_{j+1}(s) - O_j(s)) = 0,$$

since $O_{N+1}(s) = O_1(s)$. In view of the result (2.5c) a very simple form of the operator $R_z(s)$ arises if we set the parameter s equal to zero,

$$R_z = R_z(0) = \tfrac{1}{2}(\sigma_1^y\sigma_2^x - \sigma_2^x\sigma_3^y + \sigma_3^y\sigma_4^x - \sigma_4^x\sigma_5^y + \ldots - \sigma_N^x\sigma_1^y). \tag{2.15}$$

3. σ–ρ spin transformation

In this section we use the σ_j spin operators to construct a new set of spin operators ρ_j. This transformation, apart from end effects, changes the asymmetric XY hamiltonian in terms of σ spin into a symmetric but alternating XY hamiltonian in terms of ρ spin. At the same time the operator R_z above appears as essentially the total z component of ρ spin.

We may define a set of independent spin operators $\rho_j, j = 1, 2, \ldots, N$, which satisfy the same algebra as do Pauli matrices by the following equations:

$$\begin{aligned} \rho_1^x &= \sigma_1^z \sigma_2^x, \\ \rho_1^y &= \sigma_1^x, \\ \rho_1^z &= \sigma_1^y \sigma_2^x, \end{aligned} \tag{3.1a}$$

for j odd, $3 \leqslant j \leqslant N-1$,

$$\begin{aligned} \rho_j^x &= -(-1)^{(j-1)/2} \sigma_1^x \sigma_2^z \sigma_3^z \ldots \sigma_{j-1}^z \sigma_j^x \sigma_{j+1}^x, \\ \rho_j^y &= (-1)^{(j-1)/2} \sigma_1^x \sigma_2^z \sigma_3^z \ldots \sigma_j^z, \\ \rho_j^z &= \sigma_j^y \sigma_{j+1}^x, \end{aligned} \tag{3.1b}$$

for j even, $2 \leqslant j < N$,

$$\begin{aligned} \rho_j^x &= (-1)^{(j-2)/2} \sigma_1^x \sigma_2^z \sigma_3^z \ldots \sigma_j^z, \\ \rho_j^y &= (-1)^{(j-2)/2} \sigma_1^x \sigma_2^z \sigma_3^z \ldots \sigma_{j-1}^z \sigma_j^y \sigma_{j+1}^y, \\ \rho_j^z &= -\sigma_j^x \sigma_{j+1}^y, \end{aligned} \tag{3.1c}$$

and finally

$$\begin{aligned} \rho_N^x &= \mathrm{i}(-1)^{N/2} \sigma_1^y U, \\ \rho_N^y &= \sigma_N^x, \\ \rho_N^z &= -(-1)^{N/2} \sigma_N^x \sigma_1^y U, \end{aligned} \tag{3.1d}$$

where the operator U is defined by

$$U = \sigma_1^z \sigma_2^z \ldots \sigma_N^z. \tag{3.2a}$$

One may check from the definition of the ρ_j^z that we also have

$$U = \rho_1^z \rho_2^z \ldots \rho_N^z. \tag{3.2b}$$

The inverse transformation is given by

$$\begin{aligned} \sigma_1^x &= \rho_1^y \\ \sigma_1^y &= -\mathrm{i}(-1)^{N/2} \rho_N^x U, \\ \sigma_1^z &= -(-1)^{N/2} \rho_N^x \rho_1^y U, \end{aligned} \tag{3.3a}$$

for j odd, $3 \leqslant j \leqslant N-1$,

$$\begin{aligned} \sigma_j^x &= (-1)^{(N-j-1)/2} \rho_{j-1}^x \rho_j^x \rho_{j+1}^z \rho_{j+2}^z \ldots \rho_{N-1}^z \rho_N^y, \\ \sigma_j^y &= (-1)^{(N-j-1)/2} \rho_j^z \rho_{j+1}^z \ldots \rho_{N-1}^z \rho_N^y, \\ \sigma_j^z &= -\rho_{j-1}^x \rho_j^y, \end{aligned} \tag{3.3b}$$

for j even, $2 \leqslant j < N$,

$$\begin{aligned} \sigma_j^x &= (-1)^{(N-j)/2} \rho_j^z \rho_{j+1}^z \ldots \rho_{N-1}^z \rho_N^y, \\ \sigma_j^y &= -(-1)^{(N-j)/2} \rho_{j-1}^y \rho_j^y \rho_{j+1}^z \rho_{j+2}^z \ldots \rho_{N-1}^z \rho_N^y, \\ \sigma_j^z &= \rho_{j-1}^y \rho_j^x, \end{aligned} \tag{3.3c}$$

and finally,

$$\begin{aligned}\sigma_N^x &= \rho_N^y,\\ \sigma_N^y &= \rho_{N-1}^y \rho_N^z,\\ \sigma_N^z &= \rho_{N-1}^y \rho_N^x.\end{aligned} \tag{3.3d}$$

If we express $\mathscr{H}_{XY}$ in terms of ρ spin operators we find

$$\mathscr{H}_{XY} = -\tfrac{1}{4}\sum_{j=1}^{N-2}(1+(-1)^j k)(\rho_j^x\rho_{j+1}^x+\rho_j^y\rho_{j+1}^y)-\tfrac{1}{4}(1-k)(\rho_{N-1}^x\rho_N^x+(-1)^{N/2}\rho_{N-1}^y\rho_N^y U)$$
$$-\tfrac{1}{4}(1+k)((-1)^{N/2}\rho_N^x\rho_1^x U+\rho_N^y\rho_1^y). \tag{3.4}$$

In addition R_z becomes simply

$$R_z = \tfrac{1}{2}\sum_{j=1}^{N-1}\rho_j^z + \tfrac{1}{2}(-1)^{N/2}\rho_N^z U. \tag{3.5}$$

Using (3.4) and (3.5) it is simple to check again that $[\mathscr{H}_{XY}, R_z] = 0$.

4. Alternating XY model

The one-dimensional alternating Heisenberg chain has been studied in some detail before (Abraham 1969, Brooks Harris 1973). For the even simpler alternating XY model we may diagonalize the hamiltonian explicitly in terms of free fermion quasi-particle operators. Let us briefly sketch this diagonalization in order to indicate how it compares with the fermion quasi-particle diagonalization of the original asymmetric XY hamiltonian in terms of σ spin operators. Defining the projection operators

$$P_\pm = \tfrac{1}{2}(1 \pm U), \tag{4.1}$$

we may write

$$\mathscr{H}_{XY} = \mathscr{H}_{XY}^+ P_+ + \mathscr{H}_{XY}^- P_-. \tag{4.2}$$

Let us assume N is divisible by four so that $(-1)^{N/2} = 1$, and then let us look in detail at $\mathscr{H}_{XY}^+$ where no end effects complicate the diagonalization. In these circumstances we have

$$\mathscr{H}_{XY}^+ = -\tfrac{1}{4}\sum_{j=1}^{N}(1+(-1)^j k)(\rho_j^x\rho_{j+1}^x+\rho_j^y\rho_{j+1}^y), \tag{4.3}$$

with $\rho_{N+1} = \rho_1$. Make a Jordan–Wigner transformation to fermion operators by

$$\begin{aligned} d_j^\dagger &= \rho_1^z\rho_2^z\ldots\rho_{j-1}^z\rho_j^+,\\ d_j &= \rho_1^z\rho_2^z\ldots\rho_{j-1}^z\rho_j^-,\end{aligned} \tag{4.4}$$

where $d_{N+1} = -d_1$. Introduce the Fourier transform of these operators by

$$\mu_q^\dagger = N^{-1/2}\sum_{j=1}^{N} e^{iqj} d_j^\dagger, \tag{4.5a}$$

where the wavenumbers q satisfy

$$e^{iqN} = -1, \tag{4.5b}$$

or

$$q = \pm\frac{\pi}{N}, \pm\frac{3\pi}{N}, \ldots, \pm\left(\frac{N-1}{N}\right)\pi. \tag{4.5c}$$

We then obtain

$$\mathscr{H}^{+}_{XY} = \sum_{0<q} [\cos q(\mu_q^\dagger\mu_q - \mu_{q-\pi}^\dagger\mu_{q-\pi}) - \mathrm{i}k \sin q(\mu_q^\dagger\mu_{q-\pi} - \mu_{q-\pi}^\dagger\mu_q)]. \tag{4.6}$$

Introduce fermion quasi-particle operators ζ_q by a Bogoliubov–Valatin transformation (Bogoliubov 1958, Valatin 1958) for $q > 0$,

$$\begin{pmatrix} \zeta_q^\dagger \\ \zeta_{q-\pi}^\dagger \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\gamma_q & \mathrm{i}\sin\frac{1}{2}\gamma_q \\ \mathrm{i}\sin\frac{1}{2}\gamma_q & \cos\frac{1}{2}\gamma_q \end{pmatrix} \begin{pmatrix} \mu_q^\dagger \\ \mu_{q-\pi}^\dagger \end{pmatrix}, \tag{4.7}$$

where the angle γ_q is defined by

$$\cos\gamma_q = \cos q/E_q, \tag{4.8a}$$

$$\sin\gamma_q = k\sin q/E_q, \tag{4.8b}$$

and

$$E_q = (\cos^2 q + k^2\sin^2 q)^{1/2}. \tag{4.8c}$$

In the present case we will choose E_q to be the positive square root for $|q| < \frac{1}{2}\pi$ and the negative square root for $|q| > \frac{1}{2}\pi$. This choice is made in order that in the limit $k \to 0$ all results go smoothly to those for the symmetric XY model. One should note that this choice differs from that in our earlier analysis (Jones 1973). After the transformation (4.7) the hamiltonian takes the diagonal form

$$\mathscr{H}^{+}_{XY} = \sum_q E_q\zeta_q^\dagger\zeta_q, \tag{4.9}$$

where the summation is over the allowed q values (4.5c).

We may compare this result with the more usual fermion quasi-particle diagonalization of $\mathscr{H}^{+}_{XY}$ in terms of σ spin. In that procedure one makes a Jordan–Wigner transformation from the σ spin operators to fermion operators c_j and their Fourier transforms η_q with the same wavenumbers q as in (4.5c). Again introduce quasi-particle operators ξ_q by

$$\begin{pmatrix} \xi_q^\dagger \\ \xi_{-q} \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\gamma_q & -\mathrm{i}\sin\frac{1}{2}\gamma_q \\ -\mathrm{i}\sin\frac{1}{2}\gamma_q & \cos\frac{1}{2}\gamma_q \end{pmatrix} \begin{pmatrix} \eta_q^\dagger \\ \eta_{-q} \end{pmatrix}, \tag{4.10}$$

where $q > 0$ and the angle γ_q is the same as in (4.8a, b). One then finds

$$\mathscr{H}^{+}_{XY} = \sum_q E_q\xi_q^\dagger\xi_q. \tag{4.11}$$

In order to see clearly the link between these two different fermion representations let us recall from our previous work (Jones 1973) that we introduced anticommuting hermitian operators a_j and b_j which were related to the real and imaginary parts of the c_j,

$$\begin{aligned} \mathrm{i}a_j &= \mathrm{e}^{-\mathrm{i}(\pi/2)j}c_j - \mathrm{e}^{\mathrm{i}(\pi/2)j}c_j^\dagger, \\ b_j &= \mathrm{e}^{-\mathrm{i}(\pi/2)j}c_j + \mathrm{e}^{\mathrm{i}(\pi/2)j}c_j^\dagger. \end{aligned} \tag{4.12}$$

Using these operators one may check that

$$c_j^\dagger = \tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}(\pi/2)j}(b_j - \mathrm{i}a_j), \tag{4.13a}$$

while

$$d_j^\dagger = \tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}(\pi/2)j}(a_j - \mathrm{i}b_{j+1}). \tag{4.13b}$$

If one inserts (4.13*a*, *b*) in the Fourier transforms and then utilizes the definitions (4.7) and (4.10) of the quasi-particle transformations, one can derive the following connection between the ζ_q and ξ_q operators:

$$\zeta_q^\dagger = \tfrac{1}{2}[(\mathrm{e}^{-\mathrm{i}q}+\mathrm{i})\xi_q^\dagger + (\mathrm{e}^{-\mathrm{i}q}-\mathrm{i})\xi_{\pi-q}], \tag{4.14a}$$

$$\xi_q^\dagger = \tfrac{1}{2}[(\mathrm{e}^{\mathrm{i}q}-\mathrm{i})\zeta_q^\dagger - (\mathrm{e}^{\mathrm{i}q}+\mathrm{i})\zeta_{\pi-q}]. \tag{4.14b}$$

Finally we note that the operator R_z may be expressed in terms of these fermion operators by

$$R_z = -\tfrac{1}{2}N + \sum_q \zeta_q^\dagger\zeta_q, \tag{4.15a}$$

and

$$R_z = \sum_{0<q} [-\sin q(\xi_q^\dagger\xi_q - \xi_{-q}^\dagger\xi_{-q}) + \tfrac{1}{2}\mathrm{i}\cos q(\xi_q^\dagger\xi_{\pi-q}^\dagger + \xi_{-q}^\dagger\xi_{q-\pi}^\dagger + \xi_q\xi_{\pi-q} + \xi_{-q}\xi_{q-\pi})]. \tag{4.15b}$$

5. Discussion

The σ–ρ spin transformation of § 3 is important here because it gives a physical picture of the Baxter quantum number as essentially the total z component of spin in an alternating XY model. However, this spin transformation may be of interest in connection with other problems. For example, if we apply the σ–ρ transformation to the XYZ hamiltonian,

$$\mathscr{H}_{XYZ} = -\tfrac{1}{4}\sum_{j=1}^{N}[(1+\Gamma)\sigma_j^x\sigma_{j+1}^x + (1-\Gamma)\sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z], \tag{5.1}$$

we obtain

$$\begin{aligned}\mathscr{H}_{XYZ} = {} & \tfrac{1}{4}\Delta(\rho_2^x\rho_4^x + \rho_4^x\rho_6^x + \dots + \rho_{N-2}^x\rho_N^x + (-1)^{N/2}\rho_N^x\rho_2^x U)\\ & + \tfrac{1}{4}\Delta(\rho_1^y\rho_3^y + \rho_3^y\rho_5^y + \dots + \rho_{N-3}^y\rho_{N-1}^y + (-1)^{N/2}\rho_{N-1}^y\rho_1^y U)\\ & - \tfrac{1}{4}\sum_{j=1}^{N-2}(1+(-1)^j\Gamma)(\rho_j^x\rho_{j+1}^x + \rho_j^y\rho_{j+1}^y)\\ & - \tfrac{1}{4}(1-\Gamma)(\rho_{N-1}^x\rho_N^x + (-1)^{N/2}\rho_{N-1}^y\rho_N^y U) - \tfrac{1}{4}(1+\Gamma)((-1)^{N/2}\rho_N^x\rho_1^x U + \rho_N^y\rho_1^y),\end{aligned} \tag{5.2}$$

which shows that the XYZ σ spin model is equivalent to a pair of coupled ρ spin Ising chains, one chain defined on the odd lattice sites, the other chain defined on the even lattice sites, and the coupling between them of alternating strength $(1+(-1)^j\Gamma)$.

Section 2 raises an interesting question because there we obtain the Baxter operator in s dependent form, $R_z(s)$. The s dependence of $R_z(s)$, of course, reflects the s dependence of Baxter's basis vectors. In a study of the XXZ model (Jones 1974) we found that the parameter s there reflected the existence of a conserved quantity different from Baxter's quantum number. We conjectured that this situation might hold in the general XYZ model. However, in order to test this idea in the XY model it is not sufficient to know the form of $R_z(s)$ alone. Rather, using our altered definition of the basis states (2.1a) one must construct eigenstates $\Psi(s)$ of $\mathscr{H}_{XY}$ following our earlier method (Jones 1973). Then the question becomes does there exist an operator $V(s)$ such that

$$\Psi(s) = V(s)\Psi(0), \tag{5.3}$$

$$R_z(s) = V(s)R_z(0)V^{-1}(s). \tag{5.4}$$

Such an operator $V(s)$ would enable us to define an s dependent σ–ρ spin transformation.

The most important problem remaining is of course to find the operator form of the Baxter quantum number for the XYZ and eight-vertex models. It would be intriguing if such an operator were again associated with a spin transformation converting $\mathscr{H}_{XYZ}$ into a spin hamiltonian which is partially symmetric but with nearest-neighbour interactions that vary from site to site.

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References

Abraham D B 1969 *J. chem. Phys.* **51** 3795–806
Baxter R J 1972 *Ann. Phys., NY* **70** 323–37
—— 1973a *Ann. Phys., NY* **76** 1–24
—— 1973b *Ann. Phys., NY* **76** 25–47
—— 1973c *Ann. Phys., NY* **76** 48–71
Bogoliubov N N 1958 *Nuovo Cim.* **7** 794–805
Brooks Harris A 1973 *Phys. Rev.* B **7** 3166–87
Jones R B 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 928–50
—— 1974 *J. Phys.* A: *Math., Nucl. Gen.* **7** in the press
Katsura S 1962 *Phys. Rev.* **127** 1508–18
Lieb E H, Schultz T D and Mattis D C 1961 *Ann. Phys., NY* **16** 407–66
Valatin J G 1958 *Nuovo Cim.* **7** 843–57